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Radial symmetry of positive solutions for semilinear elliptic equations in the unit ball via elliptic and hyperbolic geometry

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ABSTRACT

Let $n \in \mathbb{N}$ with $n \geqslant 2$, $a \in (-1, 0) \cup (0, 1]$ and $f : (0, 1) \times (0, \infty) \to \mathbb{R}$ such that for each $u \in (0, \infty)$, $r \mapsto (1 + ar^2)^{(n+2)/2} f(r, (1 + ar^2)^{-(n-2)/2} u) : (0, 1) \to \mathbb{R}$ is nonincreasing. We show that each positive solution of

$$\Delta u + f\left(|x|, u\right) = 0 \quad \text{in } B, \qquad u = 0 \quad \text{on } \partial B$$

is radially symmetric, where B is the open unit ball in $\mathbb{R}^N$.

1. Introduction

We consider symmetry and monotonicity properties of positive solutions of the problem

$$\begin{cases} \Delta u + f\left(|x|, u\right) = 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.1}$$

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where $f:(0,1)\times(0,\infty)\to\mathbb{R}$ and B is the open unit ball in $\mathbb{R}^n$, i.e., $B=\{x\in\mathbb{R}^n:\ |x|<1\}$. Celebrated Gidas–Ni–Nirenberg's theorem [10] shows that if for each $u\in(0,\infty)$, $r\mapsto f(r,u):(0,1)\to\mathbb{R}$ is nonincreasing, then any $C^2(\overline{B})$-positive solution of (1.1) is radially symmetric. Many researchers studied such symmetry properties; see [1–5,7–11,13–24] and others. In some of them, geometry plays an important role. In [19], Naito, Nishimoto and Suzuki considered the case that $n=2$ (i.e., B is the open unit ball in $\mathbb{R}^2$) and $(1-r^2)^2 f(r,u):(0,1)\to\mathbb{R}$ is decreasing for each $u\in(0,\infty)$. Using hyperbolic geometry, they showed each positive solution of (1.1) is radially symmetric. Naito and Suzuki [20] extended their result to the case that $n\geqslant 2$ and $r\mapsto(1-r^2)^{(n+2)/2}f(r,(1-r^2)^{-(n-2)/2}u)$ is decreasing for each $r\in(0,1)$ and $u\in(0,\infty)$. Almeida, Ge and Orlandi [1] gave a similar result. (Although the arguments in [1] seem to be fine, the assumption (1.2) in [1, Theorem 1.1] is not correct.)

In this paper, we consider not only hyperbolic geometry but also elliptic geometry, and we show a symmetric result of (1.1). Since we want to treat a wide class of solutions of (1.1), we recall the definition of a strong solution in [12, p. 219]. We say a function $u\in L^1_{\mathrm{loc}}(B)$ is said to be a strong solution of (1.1) if each first and second derivative of u in the sense of distribution belongs to $L^1_{\mathrm{loc}}(B)$ and u satisfies (1.1) almost everywhere in B. Now, we show our result.

Theorem 1. *Let $n\in\mathbb{N}$ with $n\geqslant 2$, $a\in(-1,0)\cup(0,1]$ and $f:(0,1)\times(0,\infty)\to\mathbb{R}$ such that*

(i) *for each $u\in(0,\infty)$, $r\mapsto(1+ar^2)^{(n+2)/2}f(r,(1+ar^2)^{-(n-2)/2}u)$ is nonincreasing,*
(ii) *for each $r_0\in(0,1)$ and $M\in(0,\infty)$,*

$$\sup\left\{\left|\frac{f(r,u_1)-f(r,u_2)}{u_1-u_2}\right|:\ (r,u_1,u_2)\in(r_0,1)\times(0,M]^2,\ u_1\neq u_2\right\}<\infty.$$

Let $u\in W^{2,n}_{\mathrm{loc}}(B)\cap C(\overline{B})$ be a positive strong solution of (1.1)*. Then u is radially symmetric. Moreover, if $u\in C^1(B)$ then $((1+ar^2)^{(n-2)/2}u)_r<0$ for $r=|x|\in(0,1)$.*

Remark 1. For related results, we give some comments.

(i) The case $a=0$ is nothing but the Gidas–Ni–Nirenberg's theorem in [10] for the case of B. The case $a=-1$ is studied in [19,20] under the assumption that for each $u\in(0,\infty)$, $r\mapsto(1-r^2)^{(n+2)/2}f(r,(1-r^2)^{-(n-2)/2}u):(0,1)\to\mathbb{R}$ is decreasing instead of nonincreasing. The reason why nonincreasingness is not enough is that (2.5) does not hold in the case $a=-1$; see the proofs of Lemmas 2 and 3 below. We also note that two coefficient functions in (3.1) are not essentially bounded in the case of $a=-1$ and hence an additional device is needed to derive the symmetry result. For the details, see [20].
(ii) As we stated, the assumption (1.2) in [1, Theorem 1.1] is not correct. However, since the domain in [1, Theorem 1.1] is an open ball whose radius is less than 1, from the arguments in [1], we can see that the result corresponds to the case $a\in(-1,0)$ of our result.
(iii) We can apply our result to the equations

$$\begin{cases}\Delta_g u+f\big(d(x,x_0),u\big)=0 & \text{in } D,\\ u=0 & \text{on } \partial D,\end{cases}$$

where we consider hyperbolic space $\mathbb{H}^n$ or sphere $\mathbb{S}^n$, D is a geodesic ball in $\mathbb{H}^n$ or $\mathbb{S}^n$ whose center is x_0 and the radius is r_0, Δ_g is the corresponding Laplace–Beltrami operator, $f:(0,r_0)\times\mathbb{R}\to\mathbb{R}$ and d is the Riemannian distance. In order to find how to apply our result, see the proof of [8, Theorem 5], in which Gidas–Ni–Nirenberg's theorem is applied. For related results, see also [1,13,23].

Remark 2. If a positive strong solution u belongs to $W^{2,q}_{\mathrm{loc}}(B) \cap C(\overline{B})$ with some $q > n$, then by the Sobolev embedding theorem (see [12, Theorem 7.26]), u belongs to $C^1(B)$, and hence u is radially symmetric and $((1+ar^2)^{(n-2)/2}u)_r < 0$ for $r = |x| \in (0,1)$.

As an application, we consider the problem

$$\begin{cases} \Delta u + K(|x|)|u|^{\sigma-1}u = 0 & \text{in } B, \\ u = 0 & \text{on } \partial B \end{cases} \tag{1.2}$$

with $K \in C([0,1],\mathbb{R})$ and $1 < \sigma < 2^* - 1$, where $2^* = \infty$ if $n = 2$ and $2^* = 2n/(n-2)$ if $n \geqslant 3$. We say u is a weak solution of (1.2) if $u \in W^{1,2}_0(B)$ and $\int_B (\nabla u \nabla \varphi - K(|x|)|u|^{\sigma-1}u\varphi)\,dx = 0$ for all $\varphi \in C^\infty_0(B)$.

Remark 3. For problem (1.2), we note the following:

(i) The condition $\{r \in [0,1]\colon K(r) > 0\} \neq \emptyset$ is necessary and sufficient for the existence of nontrivial nonnegative weak solution. The necessity is trivial. Although the sufficiency is well known, for the reader's convenience, we give a simple and elementary proof in Appendix A.
(ii) Each weak solution belongs to $\bigcap_{q \geqslant 1} W^{2,q}(B) \cap C^1(\overline{B})$ by the elliptic regularity (see [6] and [12, Corollary 7.11, Theorems 9.13 and 9.15]), and hence it is a strong solution.
(iii) Each nontrivial nonnegative weak solution is a positive strong solution by (ii) and the strong maximum principle (see [12, Theorem 9.6]).

The following is a direct consequence of Theorem 1 and Remark 3(iii).

Corollary 1. *Let $a \in (-1,0) \cup (0,1]$. Assume that $r \mapsto (1+ar^2)^{(n+2-\sigma(n-2))/2}K(r) : (0,1) \to \mathbb{R}$ is nonincreasing. Let u be a nontrivial nonnegative weak solution of* (1.1)*. Then u is radially symmetric and $((1+ar^2)^{(n-2)/2}u)_r < 0$ for $r = |x| \in (0,1)$.*

Remark 4. Naito and Suzuki showed that the corollary above also holds in the case of $a = -1$. For the sake of completeness, contrary to [20, Theorem 1] (see also Remark 1(i) above), they showed that the decreasingness of $r \mapsto (1+ar^2)^{(n+2-\sigma(n-2))/2}K(r) : (0,1) \to \mathbb{R}$ is not needed. For the details, see [20, Corollary 1]. We note that the nonincreasingness yields $K \geqslant 0$ in the case $a = -1$.

We give examples of $K \in C([0,1],\mathbb{R})$ such that Corollary 1 is applicable but either Gidas–Ni–Nirenberg's theorem (corresponding to the case $a = 0$) or [20, Corollary 1] (corresponding to the case $a = -1$) is not applicable. Moreover, Example 1 below shows that to consider elliptic geometry (corresponding to $a \in (0,1]$) is meaningful. For the sake of simplicity, we consider the case $n \geqslant 3$. Let $a \in [-1,1]$. We note that

$$\left(1+ar^2\right)^{\frac{n+2}{2}} K(r)\left(\left(1+ar^2\right)^{-\frac{n-2}{2}}u\right)^{\sigma} = \left(1+ar^2\right)^{\frac{n+2-\sigma(n-2)}{2}} K(r)u^{\sigma}.$$

We set $b = (n+2-\sigma(n-2))/2$. Then we can find that $1 < \sigma < 2^* - 1$ is equivalent to $0 < b < 2$ and that

$$\frac{d}{dr}\left(\left(1+ar^2\right)^b K(r)\right) = \left(1+ar^2\right)^{b-1}\left(2abrK(r) + \left(1+ar^2\right)K'(r)\right)$$

for each $r \in (0,1)$ at which K is differentiable.

Example 1. Let $a = 1$, $1 < \sigma < 2^* - 1$ and $0 < k \leqslant \min\{2b/(3b+2), 4b/5\}$, where $b = (n+2-\sigma(n-2))/2$, and we set

$$K(r) = \begin{cases} -4r+1 & \text{for } 0 \leqslant r \leqslant 1/2, \\ k(r-1/2)-1 & \text{for } 1/2 \leqslant r \leqslant 1. \end{cases}$$

Then we have

$$\frac{d}{dr}\left(\left(1+ar^2\right)^b K(r)\right) \leqslant 0 \quad \text{for } r \in (0,1) \setminus \{1/2\}.$$

Hence from Corollary 1, any nontrivial nonnegative weak solution of (1.2) is radially symmetric. We note that since K is negative, increasing on $(1/2, 1)$, we cannot apply the case $a \in [-1, 0]$.

Example 2. Let $a = -1/2$, $1 < \sigma < 2^* - 1$ and $0 < k \leqslant 6b/(2b+17)$, where $b = (n+2-\sigma(n-2))/2$, and we set

$$K(r) = \begin{cases} 1-k/3 & \text{for } 0 \leqslant r \leqslant 1/3, \\ k(r-2/3)+1 & \text{for } 1/3 \leqslant r \leqslant 2/3, \\ -6(r-2/3)+1 & \text{for } 2/3 \leqslant r \leqslant 1. \end{cases}$$

Then we can see

$$\frac{d}{dr}\left(\left(1+ar^2\right)^b K(r)\right) \leqslant 0 \quad \text{for } r \in (0,1) \setminus \{1/3, 2/3\}.$$

Hence from Corollary 1, any nontrivial nonnegative weak solution of (1.2) is radially symmetric. We note that since K is positive, increasing on $(1/3, 2/3)$ and K is negative on $(5/6, 1]$, we cannot apply the case $a \in [0, 1] \cup \{-1\}$.

Next, we consider the problem

$$\begin{cases} \Delta u + f\left(|x|, u\right) = 0 & \text{in } B \setminus \{0\}, \\ u = 0 & \text{on } \partial B, \\ \lim_{x \to 0} u(x) = \infty. \end{cases} \tag{1.3}$$

Theorem 2. *Let n, a and f be as in Theorem 1. Let $u \in W^{2,n}_{\mathrm{loc}}(B \setminus \{0\}) \cap C(\overline{B} \setminus \{0\})$ be a positive strong solution of (1.3). Then u is radially symmetric. Moreover, if $u \in C^1(B \setminus \{0\})$ then $((1+ar^2)^{(n-2)/2}u)_r < 0$ for $r = |x| \in (0, 1)$.*

This paper is organized as follows. In Section 2, we give preliminaries for elliptic and hyperbolic geometry. In Sections 3 and 4, we give the proofs of Theorems 1 and 2, respectively. Finally, we give Appendices A and B.

2. Elliptic and hyperbolic geometry

As in the previous section, throughout this paper, we denote by $|\cdot|$ the standard Euclidean norm. Let $a \in [-1, 1] \setminus \{0\}$. We consider that (B, g) is a Riemannian manifold, where the metric tensor g is defined by

$$\frac{4|a|\,|dx|^2}{(1+a|x|^2)^2}. \tag{2.1}$$

For each $\lambda \in (0, 1)$, let $T_\lambda \subset B$ be a totally geodesic plane which intersects x_1-axis orthogonally at $(\lambda, 0, \ldots, 0)$. Then we can find that

$$T_\lambda = \left\{ x \in B\colon |x - e_\lambda| = \frac{1 + a\lambda^2}{2|a|\lambda} \right\}, \tag{2.2}$$

where

$$e_\lambda = \left(\frac{1 - a\lambda^2}{2(-a)\lambda}, 0, \dots, 0 \right). \tag{2.3}$$

For each $\lambda \in (0,1)$, we define $\Sigma_\lambda \subset B$ by

$$\Sigma_\lambda = \begin{cases} \{x \in B\colon |x - e_\lambda| > \frac{1+a\lambda^2}{2a\lambda}\} & \text{if } a \in (0,1], \\ \{x \in B\colon |x - e_\lambda| < \frac{1+a\lambda^2}{2(-a)\lambda}\} & \text{if } a \in [-1,0). \end{cases}$$

For each $\lambda \in (0,1)$ and $x \in \Sigma_\lambda$, there is the reflection $h_\lambda(x)$ of x with respect to T_λ in (B, g) and it is given by

$$h_\lambda(x) = e_\lambda + \left(\frac{1 + a\lambda^2}{2a\lambda} \right)^2 \frac{x - e_\lambda}{|x - e_\lambda|^2}. \tag{2.4}$$

Considering Σ_λ as a subset of the Euclidean space $\mathbb{R}^n$, we can see that h_λ is uniquely continuously extended to $\overline{\Sigma_\lambda}$. We also denote it by h_λ. Then h_λ satisfies (2.4) for all $x \in \overline{\Sigma_\lambda}$. In the case of $a \neq -1$, it holds that

$$\left| h_\lambda(x) \right| < |x| \quad \text{for each } \lambda \in (0,1) \text{ and } x \in \Sigma_\lambda \cup \operatorname{Int}_{\partial B}(\overline{\Sigma_\lambda} \cap \partial B). \tag{2.5}$$

Here, for a subset E of ∂B, we denote by $\operatorname{Int}_{\partial B} E$, the interior set of E with respect to the relative topology of ∂B. In the case of $a = -1$, it holds that

$$\left| h_\lambda(x) \right| < |x| \quad \text{for each } \lambda \in (0,1) \text{ and } x \in \Sigma_\lambda \tag{2.6}$$

and $|h_\lambda(x)| = 1$ for each $\lambda \in (0,1)$ and $x \in \overline{\Sigma_\lambda} \cap \partial B$. For the details of the facts above, see Appendix B.

3. Proof of Theorem 1

Let n, a, f and u be as in Theorem 1. As in the previous section, we consider that (B, g) is a Riemannian manifold, where the metric tensor g is defined by (2.1). We note that the Laplace–Beltrami operator on (B, g) at $x \in B$ is given by

$$\Delta_{(g,x)} = \frac{(1 + a|x|^2)^2}{4|a|} \left(\Delta - \frac{2a(n-2)}{1 + a|x|^2} \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} \right).$$

We set $v(x) = (1 + a|x|^2)^{(n-2)/2} u(x)$ for $x \in \overline{B}$. Then we have

$$\begin{aligned} \Delta_{(g,x)} v(x) &= \frac{(1 + a|x|^2)^2}{4|a|} \left(\Delta v - \frac{2a(n-2)}{1 + a|x|^2} \sum_{i=1}^n x_i \frac{\partial v}{\partial x_i} \right) \\ &= \frac{(1 + a|x|^2)^{\frac{n+2}{2}}}{4|a|} \Delta u(x) + \frac{n(n-2)a}{4|a|} \left(1 + a|x|^2\right)^{\frac{n-2}{2}} u(x) \\ &= -\frac{(1 + a|x|^2)^{\frac{n+2}{2}}}{4|a|} f\left(|x|, \left(1 + a|x|^2\right)^{-\frac{n-2}{2}} v(x)\right) + \frac{n(n-2)a}{4|a|} v(x) \end{aligned}$$

for $x \in B$. For each $\lambda \in (0,1)$, we define $w_\lambda \in W^{2,n}_{\mathrm{loc}}(\Sigma_\lambda) \cap C(\overline{\Sigma_\lambda})$ and $c_\lambda \in L^\infty(\Sigma_\lambda)$ by

$$w_\lambda(x) = v(x) - v\big(h_\lambda(x)\big) \quad \text{for } x \in \overline{\Sigma_\lambda},$$

$$c_\lambda(x) = \begin{cases} 0 & \text{for } x \in \Sigma_\lambda \text{ with } w_\lambda(x) = 0, \\ -\frac{f(|x|,(1+a|x|^2)^{-\frac{n-2}{2}} v(x)) - f(|x|,(1+a|x|^2)^{-\frac{n-2}{2}} v(h_\lambda(x)))}{(1+a|x|^2)^{-\frac{n-2}{2}} v(x) - (1+a|x|^2)^{-\frac{n-2}{2}} v(h_\lambda(x))} & \text{for } x \in \Sigma_\lambda \text{ with } w_\lambda(x) \neq 0. \end{cases}$$

We note that $0 \notin \overline{\Sigma_\lambda}$ for each $\lambda \in (0,1)$ and that the assumption (ii) in Theorem 1 yields

$$\sup_{r<\lambda<1} \operatorname*{ess\,sup}_{x \in \Sigma_\lambda} \big|c_\lambda(x)\big| < \infty \quad \text{for each } r \in (0,1).$$

Lemma 1. *For each $\lambda \in (0,1)$ and almost every $x \in \Sigma_\lambda$,*

$$-\Delta w_\lambda + \frac{2a(n-2)}{1+a|x|^2} \sum_{i=1}^n x_i \frac{\partial w_\lambda}{\partial x_i} + \frac{n(n-2)a}{(1+a|x|^2)^2} w_\lambda + c_\lambda w_\lambda \leqslant 0. \tag{3.1}$$

Proof. Let $\lambda \in (0,1)$, $x \in \Sigma_\lambda$ and set $y = h_\lambda(x)$. By the definition of h_λ, we know that $h_\lambda : (\Sigma_\lambda, g) \to (h_\lambda(\Sigma_\lambda), g)$ is Riemannian isometric. So we have $\Delta_{(g,y)} v(y) = \Delta_{(g,x)}(v(h_\lambda(x)))$. Using this equality and the assumption (i) in Theorem 1, we have

$$\begin{aligned}
0 &= \Delta_{(g,y)}\big(v(y)\big) - \frac{n(n-2)a}{4|a|} v(y) + \frac{(1+a|y|^2)^{\frac{n+2}{2}}}{4|a|} f\Big(|y|, \big(1+a|y|^2\big)^{-\frac{n-2}{2}} v(y)\Big) \\
&\quad - \Delta_{(g,x)} v(x) + \frac{n(n-2)a}{4|a|} v(x) - \frac{(1+a|x|^2)^{\frac{n+2}{2}}}{4|a|} f\Big(|x|, \big(1+a|x|^2\big)^{-\frac{n-2}{2}} v(x)\Big) \\
&= -\Delta_{(g,x)} w_\lambda(x) + \frac{n(n-2)a}{4|a|} w_\lambda(x) \\
&\quad + \frac{(1+a|h_\lambda(x)|^2)^{\frac{n+2}{2}}}{4|a|} f\Big(\big|h_\lambda(x)\big|, \big(1+a\big|h_\lambda(x)\big|^2\big)^{-\frac{n-2}{2}} v\big(h_\lambda(x)\big)\Big) \\
&\quad - \frac{(1+a|x|^2)^{\frac{n+2}{2}}}{4|a|} f\Big(|x|, \big(1+a|x|^2\big)^{-\frac{n-2}{2}} v(x)\Big) \\
&\geqslant -\Delta_{(g,x)} w_\lambda(x) + \frac{n(n-2)a}{4|a|} w_\lambda(x) + \frac{(1+a|x|^2)^{\frac{n+2}{2}}}{4|a|} f\Big(|x|, \big(1+a|x|^2\big)^{-\frac{n-2}{2}} v\big(h_\lambda(x)\big)\Big) \\
&\quad - \frac{(1+a|x|^2)^{\frac{n+2}{2}}}{4|a|} f\Big(|x|, \big(1+a|x|^2\big)^{-\frac{n-2}{2}} v(x)\Big) \\
&= -\Delta_{(g,x)} w_\lambda(x) + \frac{n(n-2)a}{4|a|} w_\lambda(x) + \frac{(1+a|x|^2)^2}{4|a|} c_\lambda(x) w_\lambda(x) \\
&= \frac{(1+a|x|^2)^2}{4|a|} \left(-\Delta w_\lambda + \frac{2a(n-2)}{1+a|x|^2} \sum_{i=1}^n x_i \frac{\partial w_\lambda}{\partial x_i} + \frac{n(n-2)a}{(1+a|x|^2)^2} w_\lambda + c_\lambda w_\lambda\right).
\end{aligned}$$

Thus we obtain (3.1). $\square$

Now, we apply the moving plane argument. We set

$$A = \left\{\lambda \in (0,1)\colon\ w_\lambda(x) < 0 \text{ for each } x \in \Sigma_\lambda\right\} \quad \text{and} \quad \mu = \inf_{\lambda \in A} \lambda. \tag{3.2}$$

Lemma 2. $A \neq \emptyset$.

Proof. Let $\lambda \in (0,1)$ be close to 1. Then we can see $|\Sigma_\lambda| \ll 1$. Since (3.1) holds for almost every $x \in \Sigma_\lambda$ and $w_\lambda(x) \leqslant 0$ for each $x \in \partial\Sigma_\lambda$, by the Alexandroff–Bakelman–Pucci inequality [12, Theorem 9.1], which is abbreviated to the ABP inequality below, we have $w_\lambda(x) \leqslant 0$ for $x \in \overline{\Sigma_\lambda}$. From (2.5), we have $w_\lambda(x) < 0$ for $x \in \mathrm{Int}_{\partial B}(\overline{\Sigma_\lambda} \cap \partial B)$. By the strong maximum principle (see [12, Theorem 9.6]), we have $w_\lambda(x) < 0$ for $x \in \Sigma_\lambda$. $\square$

Lemma 3. $\mu = 0$.

Proof. Suppose not, i.e., $\mu \in (0,1)$. Then we have $w_\mu \leqslant 0$ on $\overline{\Sigma_\mu}$. Since $w_\mu(x) < 0$ for $x \in \mathrm{Int}_{\partial B}(\overline{\Sigma_\mu} \cap \partial B)$ from (2.5), we have $\mu \in A$ by the strong maximum principle. Let G be an open set such that $\overline{G} \subset \Sigma_\mu$ and $|\Sigma_\mu \setminus G| \ll 1$. Then we have $\max_{x \in \overline{G}} w_\mu(x) < 0$. Let $0 < \varepsilon \ll 1$. Then we have $\max_{x \in \overline{G}} w_{\mu-\varepsilon}(x) < 0$ and $|\Sigma_{\mu-\varepsilon} \setminus \overline{G}| \ll 1$. We note that (3.1) holds with $\lambda = \mu - \varepsilon$. Since $w_{\mu-\varepsilon}(x) \leqslant 0$ for $x \in \partial(\Sigma_{\mu-\varepsilon} \setminus \overline{G})$ and $w_{\mu-\varepsilon}(x) < 0$ for $x \in \partial G$, by the ABP inequality and the strong maximum principle, we have $w_{\mu-\varepsilon}(x) < 0$ for $x \in \Sigma_{\mu-\varepsilon}$, which is a contradiction. Hence we have shown $\mu = 0$. $\square$

Proof of Theorem 1. From the lemmas above, we have $v(x) \leqslant v(h_0(x))$ for each $x \in B$ with $x_1 \geqslant 0$, where $h_0(x) = (-x_1, x_2, \dots, x_n)$ for $x \in B$. We note that Eq. (1.1) is invariant under rotation. So by the same arguments for the negative x_1-direction, we have $v(x) = v(h_0(x))$ for each $x \in B$. Since we can take any direction as x_1-direction, we can infer that v is radially symmetric about the origin.

Assume $u \in C^1(B)$. Since for each $\lambda \in (0,1)$, $w_\lambda < 0$ in Σ_λ and $w_\lambda = 0$ on $\partial\Sigma_\lambda \setminus \partial B$, by Hopf's lemma (see [12, Lemma 3.4] and its proof), we can infer that $v_r(|x|) < 0$ for each $r = |x| \in (0,1)$. Hence we obtain the conclusion. $\square$

4. Proof of Theorem 2

Since the arguments in the previous section almost similarly work for the proof of Theorem 2, we only give its sketch. Let n, a, f and u be as in Theorem 2. For $\lambda \in (0,1)$, we define v, Σ_λ and h_λ as before. If there is $x \in \overline{\Sigma_\lambda}$ satisfying $h_\lambda(x) = 0$, we denote such x by x_λ. If there is no $x \in \Sigma_\lambda$ with $h_\lambda(x) = 0$ (resp. no $x \in \overline{\Sigma_\lambda}$ with $h_\lambda(x) = 0$), we consider $\Sigma_\lambda \setminus \{x_\lambda\} = \Sigma_\lambda$ (resp. $\overline{\Sigma_\lambda} \setminus \{x_\lambda\} = \overline{\Sigma_\lambda}$). We define w_λ and c_λ as before. We note that $w_\lambda \in W^{2,n}_{\mathrm{loc}}(\Sigma_\lambda \setminus \{x_\lambda\}) \cap C(\overline{\Sigma_\lambda} \setminus \{x_\lambda\})$ and c_λ may not belong to $L^\infty(\Sigma_\lambda)$. However, we have

$$\sup_{r<\lambda<1} \operatorname{ess\,sup}\left\{\left|c_\lambda(x)\right|\colon\ x \in \Sigma_\lambda,\ w_\lambda(x) \geqslant -1\right\} < \infty \quad \text{for each } r \in (0,1).$$

By a similar proof of Lemma 1, we can show that (3.1) holds for each $\lambda \in (0,1)$ and almost every $x \in \Sigma_\lambda \setminus \{x_\lambda\}$. We set A and μ by (3.2). We can show $A \neq \emptyset$ by a similar proof of Lemma 2. We will show $\mu = 0$. As before, we assume that the conclusion $\mu = 0$ does not hold, i.e., we assume $\mu \in (0,1)$. In the case that there is no $x \in \overline{\Sigma_\mu}$ satisfying $h_\mu(x) = 0$, we can obtain a contradiction as before. So we consider the case $x_\mu \in \overline{\Sigma_\mu}$. For the sake of completeness, we note that $\mu \leqslant \inf\{\lambda \in (0,1)\colon$ there is no $x \in \overline{\Sigma_\lambda}$ satisfying $h_\lambda(x) = 0\}$ and if equality holds then $x_\mu = (1, 0, \dots, 0)$. Since $w_\mu \leqslant -1$ on $U \equiv \{x \in \overline{\Sigma_\mu}\colon\ |x - x_\mu| < \varepsilon\}$ with $0 < \varepsilon \ll 1$, and (3.1) holds for all $x \in \Sigma_\mu \setminus \overline{U}$, we have $w_\mu < 0$ in Σ_μ by the strong maximum principle. Thus we have $\mu \in A$. Let G be an open set such that $\overline{G} \subset \Sigma_\mu$ and $|\Sigma_\mu \setminus \overline{G}| \ll 1$. We have $\max_{x \in \overline{G}} w_\mu(x) < 0$. Let $\lambda \in (0,\mu)$ which is sufficiently close to μ, and let V be a neighborhood of x_λ such that $\overline{V} \subset \Sigma_\lambda$ and $w_\lambda \leqslant -1$ on V. We set

$H = G \cup V$. Then we have $\overline{H} \subset \Sigma_\lambda$, $\max_{\overline{H}} w_\lambda < 0$, $|\Sigma_\lambda \setminus \overline{H}| \ll 1$, and (3.1) holds for all $x \in \Sigma_\lambda \setminus \overline{H}$. By the ABP inequality and the strong maximum principle, we have $w_\lambda(x) < 0$ for all $x \in \Sigma_\lambda$, which is a contradiction. Thus we have shown $\mu = 0$. Since by similar ways as in the proof of Theorem 1, we can show that u is radially symmetric and $((1+ar^2)^{(n-2)/2}u)_r < 0$ for $r = |x| \in (0,1)$ in the case of $u \in C^1(B \setminus \{0\})$, we finish the proof.

Appendix A. Existence of a nontrivial nonnegative weak solution of (1.2)

Here, we show that if $K \in C([0,1],\mathbb{R})$ has $r \in [0,1]$ with $K(r) > 0$, then problem (1.2) has a nontrivial nonnegative weak solution. Since there is $r \in [0,1]$ with $K(r) > 0$, the set $\mathcal{A} \equiv \{u \in W_0^{1,2}(B)\colon \int_B K(|x|)|u|^{\sigma+1}\,dx > 0\}$ is nonempty. We define $J : \mathcal{A} \to \mathbb{R}$ by

$$J(u) = \frac{\int_B |\nabla u|^2\,dx}{(\int_B K(|x|)|u|^{\sigma+1}\,dx)^{2/(\sigma+1)}} \quad \text{for } u \in \mathcal{A}.$$

Then J satisfies

$$J\big(|u|\big) = J(u) \quad \text{and} \quad J(tu) = J(u) \quad \text{for each } u \in \mathcal{A} \text{ and } t > 0. \tag{A.1}$$

Let $\mathcal{L} = \{u \in W_0^{1,2}(B)\colon \int_B K(|x|)|u|^{\sigma+1}\,dx = 1\}$. We can easily see that $c \equiv \inf_{u\in\mathcal{L}} J(u) > 0$. Let $\{u_n\} \subset \mathcal{L}$ such that $J(u_n) \to c$. We may assume $\{u_n\}$ converges weakly to u in $W_0^{1,2}(B)$. Since $1 < \sigma < 2^* - 1$, by the Sobolev embedding theorem, we have $u \in \mathcal{L}$. Hence we have $J(u) = c$ by the weak lower semicontinuity of $w \mapsto \int_B |\nabla w|^2\,dx : W_0^{1,2}(B) \to \mathbb{R}$. We choose $s > 0$ such that $v \equiv s|u|$ satisfies $\int_B |\nabla v|^2\,dx = \int_B K(|x|)|v|^{\sigma+1}\,dx$. By (A.1), we have $J(v) = \min_{w\in\mathcal{A}} J(w)$. For each $\varphi \in C_0^\infty(B)$, we consider the functional $h_\varphi(t) \equiv J(v + t\varphi)$ for $t \in \mathbb{R}$. Since $J(v) = \min_{w\in\mathcal{A}} J(w)$, we have

$$0 = h'_\varphi(0) = \frac{2\int_B \nabla v \nabla\varphi\,dx\, C^{\frac{2}{\sigma+1}} - \frac{2}{\sigma+1}\int_B |\nabla v|^2\,dx\, C^{\frac{2}{\sigma+1}-1}(\sigma+1)\int_B |v|^{\sigma-1}v\varphi\,dx}{C^{\frac{4}{\sigma+1}}},$$

where $C = \int_B K(|x|)|v|^{\sigma+1}\,dx$. Hence v is a weak solution of (1.2). Since v is nontrivial and nonnegative, we have shown our assertion.

Appendix B. Proofs of (2.2)–(2.5)

We give the proofs for (2.2)–(2.5). First, we consider the case $a = 1$. We set $S^+ = \{X \in R^{n+1}\colon |X| = 1,\ X_{n+1} > 0\}$ and we define $P : S^+ \to B$ by

$$P(X_1,\dots,X_n,X_{n+1}) = \frac{1}{X_{n+1}+1}(X_1,\dots,X_n).$$

We know that $P : (S^+, |dX|^2) \to (B, g)$ is Riemannian isometric and

$$P^{-1}(x_1,\dots,x_n) = \frac{1}{1+|x|^2}\big(2x_1,\dots,2x_n,1-|x|^2\big).$$

Let $\lambda \in (0,1)$. We set $P^{-1}(\lambda,0,\dots,0) = (\cos\theta,0,\dots,0,\sin\theta)$. Then we have

$$\cos\theta = \frac{2\lambda}{1+\lambda^2} \quad \text{and} \quad \sin\theta = \frac{1-\lambda^2}{1+\lambda^2}.$$

Since $P^{-1}(T_\lambda)$ must be $\{(X_1,\dots,X_n,X_{n+1})\in S^+\colon X_{n+1}=X_1\tan\theta\}$, we can get

$$T_\lambda=\left\{x\in B\colon (x_1+\tan\theta)^2+x_2^2+\cdots+x_n^2=1+\tan^2\theta\right\}.$$

Hence we get (2.3) and (2.2). Let $x\in\Sigma_\lambda$ and set $(X_1,\dots,X_n,X_{n+1})=P^{-1}(x)$ and $(Y_1,\dots,Y_n,Y_{n+1})=P^{-1}(h_\lambda(x))$. Then we can easily see

$$\begin{pmatrix} Y_1\\ Y_2\\ \vdots\\ Y_n\\ Y_{n+1}\end{pmatrix}=R(\theta)\,Q\,R(-\theta)\begin{pmatrix} X_1\\ X_2\\ \vdots\\ X_n\\ X_{n+1}\end{pmatrix}=\begin{pmatrix} X_1\cos 2\theta+X_{n+1}\sin 2\theta\\ X_2\\ \vdots\\ X_n\\ X_1\sin 2\theta-X_{n+1}\cos 2\theta\end{pmatrix},$$

where

$$R(\theta)=\begin{pmatrix}\cos\theta & & & & -\sin\theta\\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ \sin\theta & & & & \cos\theta\end{pmatrix}\quad\text{and}\quad Q=\begin{pmatrix}1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & -1\end{pmatrix}.$$

Setting $t=\tan\theta$, we can find

$$\begin{aligned}
h_\lambda(x)&=\frac{(x_1(\cos^2\theta-\sin^2\theta)+(1-|x|^2)\sin\theta\cos\theta,x_2,\dots,x_n)}{\cos^2\theta((x_1+\tan\theta)^2+x_2^2+\cdots+x_n^2)}\\
&=\frac{(x_1(1-t^2)+(1-|x|^2)t,(1+t^2)x_2,\dots,(1+t^2)x_n)}{|x-e_\lambda|^2}\\
&=\frac{-(1-|x|^2+2x\cdot e_\lambda)e_\lambda+(1+|e_\lambda|^2)x}{|x-e_\lambda|^2}\\
&=e_\lambda+\left(\frac{1+\lambda^2}{2\lambda}\right)^2\frac{x-e_\lambda}{|x-e_\lambda|^2},
\end{aligned}$$

where $x\cdot e_\lambda$ is the standard Euclidean inner product of x and e_λ. Thus we have shown (2.4). Since $((1+\lambda^2)/(2\lambda))^2=|e_\lambda|^2+1$ and $-2e_\lambda\cdot(x-e_\lambda)=|x-e_\lambda|^2+|e_\lambda|^2-|x|^2$, we also have

$$\begin{aligned}
|x|^2-\left|h_\lambda(x)\right|^2&=\frac{|x|^2|x-e_\lambda|^4-\left||x-e_\lambda|^2e_\lambda+(|e_\lambda|^2+1)(x-e_\lambda)\right|^2}{|x-e_\lambda|^4}\\
&=\frac{|x|^2|x-e_\lambda|^2-|x-e_\lambda|^2|e_\lambda|^2-2(|e_\lambda|^2+1)e_\lambda\cdot(x-e_\lambda)-(|e_\lambda|^2+1)^2}{|x-e_\lambda|^2}\\
&=\frac{1+|x|^2}{|x-e_\lambda|^2}\left(|x-e_\lambda|^2-|e_\lambda|^2-1\right)=\frac{1+|x|^2}{|x-e_\lambda|^2}\left(|x-e_\lambda|^2-\left(\frac{1+\lambda^2}{2\lambda}\right)^2\right)
\end{aligned}$$

for $x\in\overline{\Sigma_\lambda}$, which shows (2.5). Now, we consider the case $a\in(0,1)$. We note that the mapping η defined by

$$x\mapsto\eta(x)=\sqrt{a}x:\left(\left\{x\in\mathbb{R}^n\colon |x|<\frac{1}{\sqrt{a}}\right\},\frac{4|a|\,|dx|^2}{(1+a|x|^2)^2}\right)\to\left(B,\frac{4\,|dx|^2}{(1+|x|^2)^2}\right)$$

is Riemannian isometric and that (B, g) is a submanifold of $(\{x \in \mathbb{R}^n\colon\ |x| < 1/\sqrt{a}\}, 4|a|\,|dx|^2/(1 + a|x|^2)^2)$. Let $\lambda \in (0,1)$. Since $\eta((\lambda, 0, \ldots, 0)) = (\sqrt{a}\lambda, 0, \ldots, 0)$, we have

$$e_\lambda = \eta^{-1}\left(\left(\frac{1-(\sqrt{a}\lambda)^2}{2(-\sqrt{a}\lambda)}, 0, \ldots, 0\right)\right) = \left(\frac{1-a\lambda^2}{2(-a)\lambda}, 0, \ldots, 0\right),$$

which shows (2.3). By similar arguments, we can obtain (2.2), (2.4) and (2.5).

In the case of $a = -1$, (2.2)–(2.4) and (2.6) are given in [19,20]. We note that by a similar calculation as in the case of $a = 1$, we have

$$|x|^2 - \left|h_\lambda(x)\right|^2 = \frac{1-|x|^2}{|x-e_\lambda|^2}\left(\left(\frac{1-\lambda^2}{2\lambda}\right)^2 - |x-e_\lambda|^2\right) \quad \text{for } x \in \overline{\Sigma_\lambda},$$

which shows that (2.6) holds but (2.5) does not hold in the case of $a = -1$. For $a \in (-1, 0)$, the mapping

$$x \mapsto \sqrt{-a}x : \left(\left\{x \in \mathbb{R}^n\colon\ |x| < \frac{1}{\sqrt{-a}}\right\}, \frac{4|a|\,|dx|^2}{(1+a|x|^2)^2}\right) \to \left(B, \frac{4\,|dx|^2}{(1-|x|^2)^2}\right)$$

is Riemannian isometric and (B, g) is a submanifold of $(\{x \in \mathbb{R}^n\colon\ |x| < 1/\sqrt{-a}\}, 4|a|\,|dx|^2/(1+a|x|^2)^2)$. By similar arguments as above, we can show that (2.2)–(2.5) hold for $a \in (-1, 0)$.

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